Handout 1. Hitting springs with hammers

Reference: Edwards & Penney, Elementary Differential Equations with Boundary Value Problems, Chapters 2, 4.

We have an object with mass m attached to a linear spring with spring constant $k > 0$ and subject to linear damping with constant $c > 0$. Its position is denoted by $y(t)$. We subject it to a known external force $F(t)$. We also know its initial position y_0 and velocity v_0. Then we have

$$my'' + cy' + ky = F(t), \; y(0) = y_0, \; y'(0) = v_0$$

We assume that the reader is familiar with the (real or complex) solutions of this equation in the homogeneous case $F = 0$ and with the method of undetermined coefficients for the case where F is of the form $p(t) e^{kt} \cos \omega t$ or $p(t) e^{kt} \sin \omega t$. In the following we'll mostly concentrate on the case where $c = 0$ for simplicity.

Let

$$H(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

Here H is for "Heaviside", but we'll call it the *unit step function.* (We reserve the letter u for later use.)

Note: setting $H(0) = 0$ (and not 1, or even 1/2, which is sometimes a good idea) in this definition is entirely unimportant at this point; the value of $H(0)$ will rarely matter to us. This will be a theme throughout: individual values of functions are often not as important as one is led to believe in high school or introductory college mathematics. We'll see why presently. We use H_a to denote the shifted step

$$H_a(t) = H(t - a)$$

that turns on at $t = a$ and we also define the *rectangular pulse*

$$R_{a,b}(t) = H_a(t) - H_b(t) = \begin{cases} 0 & t \leq a, \; t > b \\ 1 & a < t \leq b \end{cases}$$

1. *The simple hammer, part 1.* Let $m, \omega, a, J, \Delta t > 0$, with Δt thought of as small, and consider the IVP

   $$my'' + m\omega^2 y = F(t), \; y(0) = 0, \; y'(0) = 0$$

   where

   $$F(t) = \frac{J}{\Delta t} R_{a,a+\Delta t}(t)$$

   (a) What are the SI units of all of the quantities $m, \omega, a, J, \Delta t$? What is the spring constant equal to in this problem?

   (b) What is $\int_0^\infty F(t) \; dt$? What does that measure physically? Are the units right?

   (c) What is $y(t)$ for $t < a$?

   (d) The discontinuity of F at a might give us pause, appearing in the context of a differential equation. But is going from F to y more like differentiating or integrating? How many times? (Hint: what if ω is very small, so it can be ignored?) Can we integrate discontinuous functions? If we integrate a step function the appropriate number of times, will it be continuous? Differentiable? Twice differentiable?

   (e) Guided by the previous part, let's solve the problem in three pieces, and declare that the pieces should fit together to be a differentiable function.

      i. Based on that, write down the IVP that we need to solve in the interval $[a, a + \Delta t]$ and solve it. Hint: it will be prettier if you write everything as a function of $t - a$, the time since the start of the forcing.

      ii. Use the value and derivative of the solution $y(t)$ that you obtained in the previous part as your initial conditions for the IVP for y on $[a + \Delta t, \infty)$. (Similar hint as before.) Show that you can write the solution on this interval in the form

      $$y(t) = (\text{some constant}) (\cos \omega (t - (a + \Delta t)) - \cos \omega (t - a))$$

      iii. Write the full solution (for all $t \geq 0$) in two ways:

A. as a piecewise function;

B. as a function of the form

$$(\text{something})\, H_a(t) + (\text{something})\, H_{a+\Delta t}(t)$$

iv. Graph your solution (in all three intervals) explicitly in the case $m = 1$ kg, $\omega = 2\ \mathrm{s}^{-1}$, $J = 4$ N s, $a = 3$ s, $\Delta t = 10$ s. Do the three parts link up correctly? What is qualitatively different between the intervals $[3, 13]$ and $[13, \infty)$?

v. Graph your solution (in all three intervals) explicitly in the case $m = 1$ kg, $\omega = 2\ \mathrm{s}^{-1}$, $J = 4$ N s, $a = 3$ s, $\Delta t = 0.1$ s. How does this differ from the previous graph? Does the solution in the interval $[3.1, \infty)$ look more like a shifted cosine or a shifted sine?

(f) We'd like to see a simple reason why the solution in $[a + \Delta t, \infty)$ is of the form you found in (e)(ii) above. So let's solve the problem in another way, one which will bring in a big theme.

i. Find the solution to

$$my'' + m\omega^2 y = \frac{J}{\Delta t} H_a(t)$$

(Hint: this should require no additional work!)

ii. Find the solution to

$$my'' + m\omega^2 y = -\frac{J}{\Delta t} H_{a+\Delta t}(t)$$

(Same hint!)

iii. Find the solution to

$$my'' + m\omega^2 y = \frac{J}{\Delta t}\left(H_a(t) - H_{a+\Delta t}(t)\right)$$

(Hint: here's where our big theme comes in: *linearity*.)

Note that this method breaks up the forcing data "vertically" (along the y-axis) by seeing that one function is a sum of two simpler functions, whereas the piecewise method breaks up the data "horizontally" (along the x-axis). We can see that sometimes (not always) the vertical method is far simpler. That's another big theme, one that is crucial in the definition of Lebesgue integration.

2. *The simple hammer, part 2.*

(a) Temporarily denote the solution you found in 1(e)(ii) by $y_a(t)$ to emphasize its dependence on a. Show that for some other time $b > 0$, $y_b(t) = y_b(t - (b - a))$. Why does this make sense physically? How is this a symmetry principle? (That will be another big theme.)

(b) Suppose we want to solve

$$my'' + m\omega^2 y = F(t),\ y(0) = 0,\ y'(0) = 0$$

where

$$F(t) = \frac{J_1}{\Delta t} R_{a,a+\Delta t}(t) + \frac{J_2}{\Delta t} R_{b,b+\Delta t}(t)$$

with $b \geq a + \Delta t$. Do we have to do this as a five-part piecewise problem, or is there a much better way? (Hint: use another big theme, already introduced.) Does it matter if $b = a + \Delta t$? If $b < a + \Delta t$?

(c) Suppose we want to solve

$$my'' + m\omega^2 y = F(t),\ y(0) = 0,\ y'(0) = 0$$

with

$$F(t) = \sum_{n=1}^{N} \frac{J_n}{\Delta t}\left(R_{a_n, a_{n+1}}(t)\right)$$

where $a_n = a + n\Delta t$. How could we use what we've done to write down a solution easily? (Same hint!)

(d) Given an arbitrary forcing function F, could we approximate it by a function of the form in the previous part? What does this remind you of from single-variable calculus?

Note: we would like to say that we can approximate a general F *arbitrarily well* by a finite sum of step functions as in part (h), and that in the limit, we could recover the solution y corresponding to F as an appropriate *limit* of solutions to problems of the form in part (h). That brings us to another theme: *convergence of sequences of functions.* This can seem like a very theoretical topic (and it can be, and a very powerful one), but it is always tied (in principle) to concrete, practical approximation schemes. For in fact, we might not know an analytic expression for the forcing function F, and we may only know samples of its values. So an expression as in part (h) might be as good an approximation to F as we can get.

3. *The simple hammer, part 3.* We are actually most interested in the case where Δt is very small, so that the impulse is delivered almost instantaneously (hence the phrase "hitting with a hammer"). In particular, we don't care about the particular value of Δt, and we seek a simple answer that is independent of Δt. This is a classic case in which we pass to a limit—the point of a limit is often to idealize and simplify a situation that has a very small or very large parameter whose detailed value is unimportant. (If we are mistaken, and its detailed value is important, then the limit will not exist.)

(a) Consider again the solution $y(t)$ to the original problem in part 1. Hold a, J fixed and let $\Delta t \to 0$. (Hint: think about the definition of the derivative, or use L'Hopital's rule.) What is the resulting function? Call it y_{ham}.

(b) Is y_{ham} continuous? Is it differentiable? Is it twice differentiable? What is $\lim_{t\to a^+} y'(t) - \lim_{t\to a^-} y'(t)$?

(c) Given that physically, we are transferring momentum J to an object with mass m "instantaneously", are the answers to (b) reasonable? Explain briefly.

NOTE: It's a bit dubious to claim that y_{ham} is a solution to a differential equation when it does not have all of the derivatives mentioned in the equation. It's also not clear what the right hand side $F(t)$ is once we pass to the limit where $\Delta t \to 0$. It would have to be an infinitely narrow and tall spike, which isn't meaningful as an ordinary function. However this idealized solution is far too useful to give up on grounds of squeamishness, and we already (in part 1) had a solution that wasn't twice differentiable, so maybe we shouldn't be too picky.

We will see various ways to usefully weaken the requirement that a function solve a differential equation, all based on *local weighted averaging.* I'll introduce our first version of that idea in the next lecture.